Pacific Hills
Wisdom & Knowledge in Christ

Senior School Examination

2016

HSC TRIAL
EXAMINATION

Mathematics Extension 2

General Instructions	**Total Marks 100**
- Reading Time- 5 minutes - Working Time - 3 hours - Write using a blue or black pen - Board approved calculators may be used - A Standard Integrals Sheet is provided at the back of this paper which may be detached and used throughout the paper. - Marks may be deducted for careless, untidy, or badly arranged work	Section I **10 marks** - Attempt Questions 1-10 - Answer on the Multiple Choice answer sheet provided. This sheet should be detached. - Allow about 15 minutes for this section. Section II **90 marks** - Attempt questions 11 – 16 - Answer in the booklets provided. Start a new booklet for each question. Allow about 2 hours & 45 minutes

Student Number: ___________________________________

Section 1

10 marks Attempt Questions 1 – 10

Allow about 15 minutes for this section Use the multiple-choice answer sheet

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1 An object rotates at 40 rpm and is moving at 30 m/s. The radius of the motion is

(A) 1.33 m

(B) 6.37 m

(C) 7.16 m

(D) 20m

2 The eccentricity of the hyperbola $4x^2 - 25y^2 = 9$ is?

(A) $\dfrac{\sqrt{21}}{5}$

(B) $\dfrac{\sqrt{29}}{5}$

(C) $\dfrac{\sqrt{21}}{2}$

(D) $\dfrac{\sqrt{29}}{2}$

3 Let $z = 3 - i$. What is the value of $\overline{iz}$?

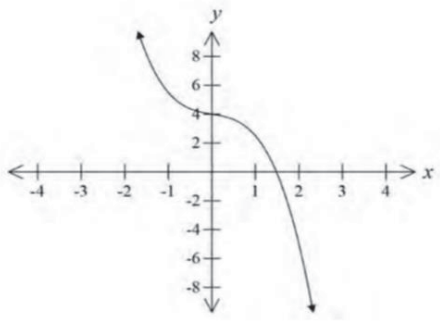
(A) $-1-3i$

(B) $-1+3i$

(C) $1-3i$

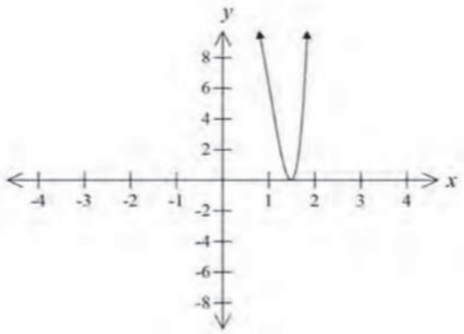
(D) $1+3i$.

4 The diagram below shows the graph of the function $y = f(x)$.

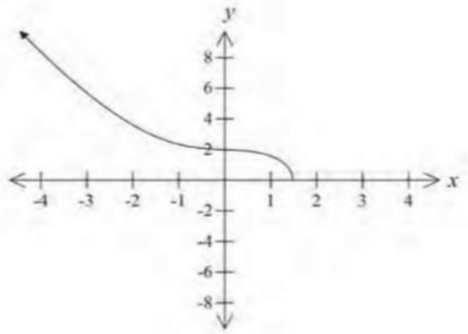


Which diagram represents the graph of $y^2 = f(x)$?

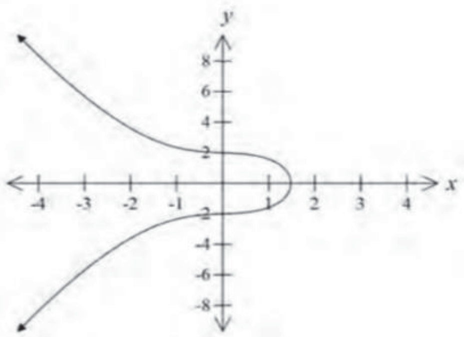
(A)



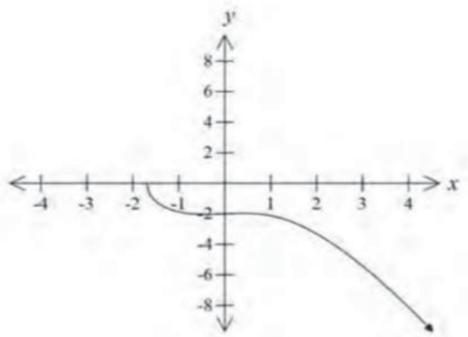
(B)



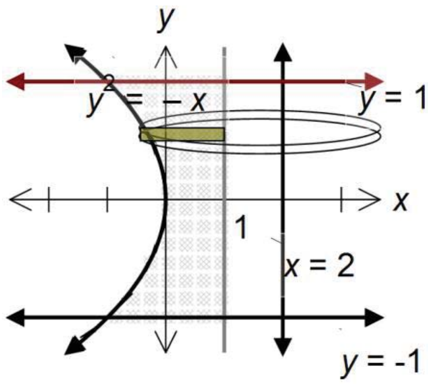
(C)



(D)



5 The region bounded by the lines $x = 1$, $y = 1$, $y = -1$ and the curve $x = -y^2$.
The region is rotated through 360^o about the line $x = 2$ to form a solid.
What is the correct expression for the volume of the solid?



(A) $V = \int_{-1}^{1} \pi\left(y^4 - 4y^2 + 3\right)dy$

(B) $V = \int_{-1}^{1} \pi\left(y^4 + 4y^2 + 3\right)dy$

(C) $V = \int_{-1}^{1} \pi\left(y^4 - 4y^2 + 4\right)dy$

(D) $V = \int_{-1}^{1} \pi\left(y^4 + 4y^2 + 4\right)dy$

6 A particle of mass m falls from rest under gravity and the resistance to the motion is mkv^2 where v is its speed and k is a positive constant. Which of the following is the correct expression for the square of the velocity, where x is the distance fallen?

.

(A) $v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$

(B) $v^2 = \frac{g}{k}\left(1 + e^{-2kx}\right)$

(C) $v^2 = \frac{g}{k}\left(1 - e^{2kx}\right)$

(D) $v^2 = \frac{g}{k}\left(1 + e^{2kx}\right)$

7 Which of these ellipses has foci $(0, \pm 3)$?

(A) $8x^2 + y^2 = 8$

(B) $5x^2 + 4y^2 = 20$

(C) $16x^2 + 25y^2 = 400$

(D) $25x^2 + 16y^2 = 400$

8 The polynomial equation $x^3 - 2x^2 + 3 = 0$ has roots α, β, and γ
What is the value of $\alpha^3 + \beta^3 + \gamma^3$?

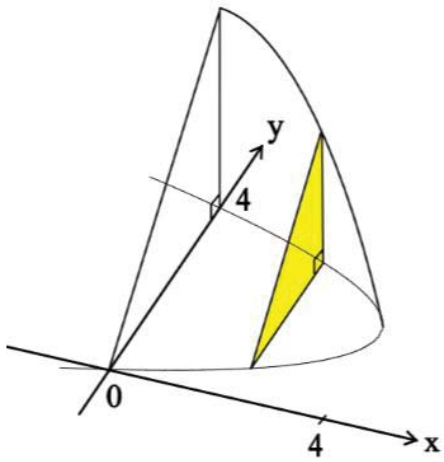
(A) -2

(B) -1

(C) -8

(D) 8

9 The base of a solid is the region bounded by the parabola $x = 4y - y^2$ and the y axis. Vertical cross sections are right angled isosceles triangles perpendicular to the x axis as shown.



Which integral represents the volume of this solid?

(A) $\int_0^4 2\sqrt{4-x}\,dx$ 1

(B) $\int_0^4 \pi(4-x)\,dx$

(C) $\int_0^4 (8-2x)\,dx$

(D) $\int_0^4 (16-4x)\,dx$

10 The points the hyperbola $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ lie on the same branch of the hyperbola $xy = c^2\,(p \neq q)$. The tangents at P and Q meet at the point T.
What is the equation of the normal to the hyperbola at P?

(A) $p^2x - py + c - cp^4 = 0$

(B) $p^3x - py + c - cp^4 = 0$

(C) $x + p^2y - 2c = 0$

(D) $x + p^2y - 2cp = 0$

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) If $z=(1-i)^{-1}$

(i) Express $\overline{z}$ in modulus argument form **2**

(ii) If $(\overline{z})^{13}=a+ib$, where a and b are real numbers find the values of a and b. **2**

(b) Find the Cartesian equation of the locus of a point P which represents the complex number z where $|z-2i|=|z|$ **2**

(c) Sketch the region in the complex plane where $\Re e\left[(2-3i)z\right]<12$ **2**

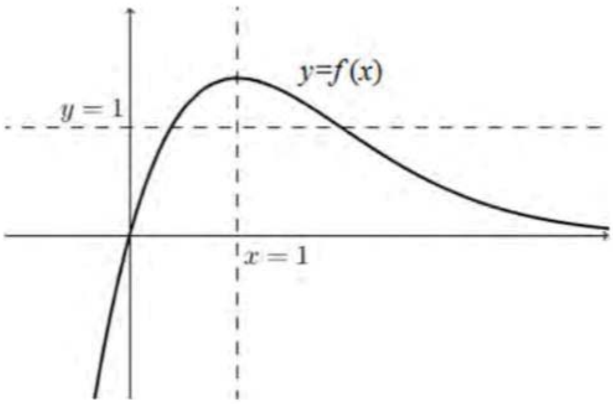
(d) The polynomial equation $x^3-3x^2-x+2=0$ has roots α, β, and γ.
Find a polynomial equation that has roots $2\alpha+\beta+\gamma$, $\alpha+2\beta+\gamma$ and $\alpha+\beta+2\gamma$? **3**

(e) i) Express $\dfrac{x^2+x+2}{(x^2+1)(x+1)}$ in the form $\dfrac{Ax+B}{x^2+1}+\dfrac{C}{x+1}$, **2**
where A, B, and C are constants

ii) Hence find $\displaystyle\int\frac{x^2+x+2}{(x^2+1)(x+1)}\,dx$ **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a)



Using four separate graphs sketch:

i) $y = f'(x)$ **2**

ii) $|y| = f(x)$ **2**

iii) $y = \dfrac{1}{f(x)}$ **2**

iv) $y = 3^{f(x)}$ **2**

(b) Evaluate $\displaystyle\int_4^7 \frac{dx}{x^2 - 8x + 19}$ **3**

(c) Let $f(x) = \dfrac{x^3 + 1}{x}$

i) Show that $\lim\limits_{x \to \pm\infty} \left[f(x) - x^2 \right] = 0$ **1**

ii) Part (i) shows the graph of $y = f(x)$ is asymptotic to the parabola $y = x^2$. **3**
Use this fact to help sketch the graph of $y = f(x)$.

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) If ω is the root of $z^5 - 1 = 0$ with the smallest positive argument, find the real quadratic equation with roots $\omega + \omega^4$ and $\omega^2 + \omega^3$. **3**

(b) Given the polynomial $P(x) = x^3 + x^2 + mx + n$ where m and n are real numbers:

(i) If $(1-2i)$ is a zero of $P(x)$, factorise $P(x)$ into complex linear factors. **2**

.

(ii) Find the values of m and n **2**

(c) i) An ellipse has major and minor axes of length 12 and 8 respectively. Write a possible equation of this ellipse. **1**

ii) A solid has the elliptical base of (i). Sections perpendicular to its base and parallel to its minor axis, are semi-circles. Find the volume of the solid **3**

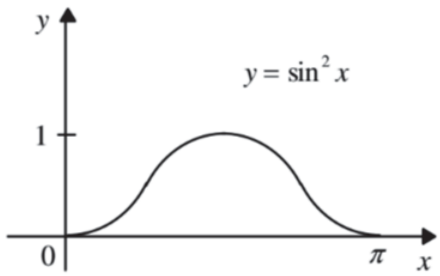
(d) i) Let $P(x)$ be a degree four polynomial with a zero of multiplicity three. Show that $P'(x)$ has a zero of multiplicity two.. **2**

ii) Hence find all the zeros of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a zero of multiplicity tree. **2**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) i) Given that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, show that $\int_0^{\pi} (x\cos 2x)\, dx = 0$ **2**

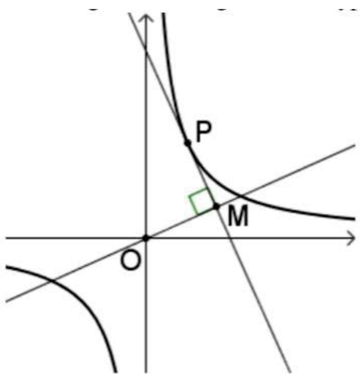
ii)



The area bounded by the curve $y = \sin^2 x$ and the x-axis between $x = 0$ and $x = \pi$ is rotated through one revolution about the y-axis.
By considering the limiting sum of the volumes of cylindrical shells find the volume of this solid **2**

(b) $P\left(t, \frac{1}{t}\right)$ is a variable point on the rectangular hyperbola $xy = 1$.

M is the foot of the perpendicular from the origin to the tangent to the hyperbola at P.



i) Show that the equation of the tangent to the hyperbola at P has equation

$$x + t^2 y = 2t$$ **2**

ii) Find the equation of OM. **1**

iii) Show that the equation of the locus of M as P varies is

$$x^4 + 2x^2y^2 - 4xy + y^4 = 0$$

and indicate any restrictions on the values of x and y. **3**

(c) A particle is fired vertically upwards with initial velocity V metres per second, and is subject to both constant gravity and air resistance proportional to speed, so that the equation of motion is given by
$\ddot{x} = -g - kv,$ where $k > 0$ is a constant, v is the velocity and g is acceleration due to gravity.

By using $\ddot{x} = v\frac{dv}{dx}$ and integrating prove that the projectile reaches a maximum height H

given by: $H = \frac{V}{k} - \frac{g}{k^2}\ln\left(1 + \frac{kV}{g}\right)$ **5**

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Use integration by parts to evaluate $\int_1^e \left(x^7 \log_e x\right)dx$ **3**

(b) i) On the same diagram sketch the graphs of

$$E_1 : \frac{x^2}{4}+\frac{y^2}{3}=1 \text{ and } E_2 : \frac{x^2}{16}+\frac{y^2}{12}=1$$ **2**

showing clearly the intercepts on the axes.
Find the coordinates of the foci and the equations of the directrices of the ellipse E_1

ii) $P\left(2\cos p, \sqrt{3}\sin p\right)$, where $0 < p < \frac{\pi}{2}$ is a point on ellipse E_1.

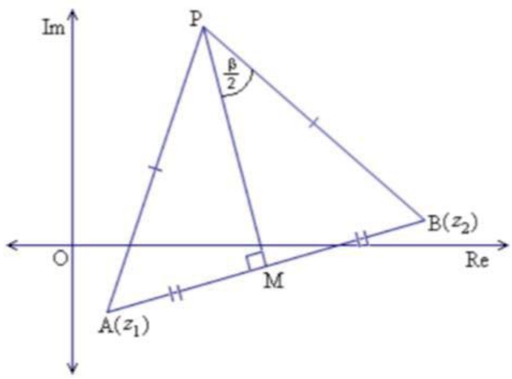
Use differentiation to show that the tangent to the ellipse E_1 at P has equation

$$\frac{x\cos p}{2}+\frac{y\sin p}{\sqrt{3}}=1$$ **2**

iii) The tangent to the ellipse E_1 at P meets ellipse E_2 at the points $Q\left(4\cos q, 2\sqrt{3}\sin q\right)$ and $R\left(4\cos r, 2\sqrt{3}\sin r\right)$
where $-\pi < q < \pi$ and $-\pi < r < \pi$.

Show that q and r differ by $\frac{2\pi}{3}$ **2**

(c)



i) Find the complex number represented by

α) $\overrightarrow{AM}$ **1**

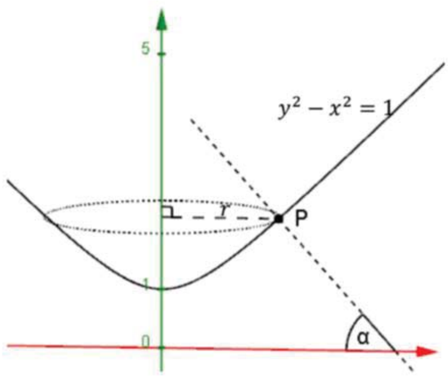
β) $\overrightarrow{MP}$ **2**

ii) Hence show that P represents the complex number

$$\frac{1}{2}\left(1-i\cot\frac{\beta}{2}\right)z_1+\frac{1}{2}\left(1+i\cot\frac{\beta}{2}\right)z_2$$ **3**

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) A bowl is formed by rotating the hyperbola $y^2 - x^2 = 1$ for $1 \le y \le 5$ through 180^o. Sometime later, a particle P of mass m moves around the inner surface of the bowl in a horizontal circle with constant angular velocity ω .



(i) Show that if the radius of the circle in which P moves is r, then the normal to the surface at P makes an angle α with the horizontal as shown in the diagram where **2**

$$\tan\alpha = \frac{\sqrt{1+r^2}}{r} .$$

(ii) Draw a diagram showing the forces on P. **1**

(iii) Find the expression for the radius r of the circle of motion and the magnitude of the reaction force between the surface and the particle in terms of m, g and ω. **3**

(iv) Find the values of ω for which the described motion of P is possible **1**

(b) Let $I_n = \int_1^e (1-\ln x)^n \, dx$ where $n = 0, 1, 2, ...$

(i) Show that $I_n = -1 + nI_{n-1}$ where $n = 0, 1, 2, ...$ **2**

(ii) Hence evaluate $\int_1^e (1-\ln x)^3 \, dx$ **2**

(iii) Show that $\frac{I_n}{n!} = e - \sum_{r=0}^{n} \frac{1}{r!}$ where $n = 1, 2, 3, ...$ **2**

(iv) Show that $0 \le I_n \le e - 1$ **1**

(v) Deduce that $\lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{r!} = e$ **1**

End of paper.

Student Number

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2016

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Multiple-Choice Answer Sheet

Select the alternative A, B, C, or D that best answers the question by placing a **X** in the box.

	A	B	C	D
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

Mathematics

Factorisation

$a^2 - b^2 = (a+b)(a-b)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Angle sum of a polygon

$S = (n-2) \times 180°$

Equation of a circle

$(x-h)^2 + (y-k)^2 = r^2$

Trigonometric ratios and identities

$\sin\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}}$

$\cos\theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}}$

$\tan\theta = \dfrac{\text{opposite side}}{\text{adjacent side}}$

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

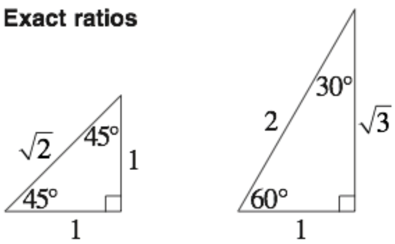
$\sec\theta = \dfrac{1}{\cos\theta}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\sin^2\theta + \cos^2\theta = 1$

Exact ratios



Sine rule

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine rule

$c^2 = a^2 + b^2 - 2ab\cos C$

Area of a triangle

$\text{Area} = \dfrac{1}{2}ab\sin C$

Distance between two points

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Perpendicular distance of a point from a line

$d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

Slope (gradient) of a line

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Point-gradient form of the equation of a line

$y - y_1 = m(x - x_1)$

*n*th term of an arithmetic series

$T_n = a + (n-1)d$

Sum to *n* terms of an arithmetic series

$S_n = \dfrac{n}{2}[2a + (n-1)d]$ or $S_n = \dfrac{n}{2}(a + l)$

*n*th term of a geometric series

$T_n = ar^{n-1}$

Sum to *n* terms of a geometric series

$S_n = \dfrac{a(r^n - 1)}{r - 1}$ or $S_n = \dfrac{a(1 - r^n)}{1 - r}$

Limiting sum of a geometric series

$S = \dfrac{a}{1-r}$

Compound interest

$A_n = P\left(1 + \dfrac{r}{100}\right)^n$

Mathematics (continued)

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\frac{dy}{dx} = F'(u)\frac{du}{dx}$

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\frac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\frac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\frac{dy}{dx} = f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$\alpha + \beta = -\frac{b}{a}$ $\qquad$ $\alpha\beta = \frac{c}{a}$

Equation of a parabola

$$(x-h)^2 = \pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Logarithms – change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Angle measure

$180° = \pi$ radians

Length of an arc

$l = r\theta$

Area of a sector

$\text{Area} = \frac{1}{2}r^2\theta$

Mathematics Extension 1

Angle sum identities

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

***t* formulae**

If $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

General solution of trigonometric equations

$\sin\theta = a, \qquad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a, \qquad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a, \qquad \theta = n\pi + \tan^{-1} a$

Division of an interval in a given ratio

$$\left(\frac{mx_2 + nx_1}{m+n}, \ \frac{my_2 + ny_1}{m+n}\right)$$

Parametric representation of a parabola

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $\left(2at, at^2\right)$,

tangent: $y = tx - at^2$

normal: $x + ty = at^3 + 2at$

At $\left(x_1, y_1\right)$,

tangent: $xx_1 = 2a\left(y + y_1\right)$

normal: $y - y_1 = -\frac{2a}{x_1}\left(x - x_1\right)$

Chord of contact from $\left(x_0, y_0\right)$: $xx_0 = 2a\left(y + y_0\right)$

Acceleration

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Simple harmonic motion

$$x = b + a\cos\left(nt + \alpha\right)$$

$$\ddot{x} = -n^2\left(x - b\right)$$

Further integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Sum and product of roots of a cubic equation

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Estimation of roots of a polynomial equation

Newton's method

$$x_2 = x_1 - \frac{f\left(x_1\right)}{f'\left(x_1\right)}$$

Binomial theorem

$$\left(a+b\right)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

Solutions

	A	B	C	D
1			X	
2		X		
3			X	
4			X	
5		X		
6	X			
7				X
8		X		
9			X	
10		X		

Q1 $v = r\omega \;\; 40revs = \dfrac{40\times 2\pi}{60} = \dfrac{4\pi}{3}$

$r \approx 30 \div \dfrac{4\pi}{3} \approx 7 \qquad \therefore C$

Q2 $4x^2 - 25y^2 = 9 \Rightarrow a^2 = \dfrac{9}{4},\ b^2 = \dfrac{9}{25}$

$b^2 = a^2\left(e^2 - 1\right) \Rightarrow \dfrac{9}{25} = \dfrac{9}{4}\left(e^2 - 1\right)$

$4 = 25e^2 - 25 \Rightarrow e^2 = \dfrac{29}{25} \Rightarrow e = \dfrac{\sqrt{29}}{5} \therefore B$

Q3 $z = 3 - i \Rightarrow iz = 1 + 3i \Rightarrow \overline{iz} = 1 - 3i \therefore C$

Q4 $y = \pm\sqrt{f(x)}$ only one graph has symmetry about x-axis

$\therefore$ C

Q5 $r = 2 - x \Rightarrow V = \pi\int_{-1}^{1}\left[(2-x)^2 - 1^2\right]dy = \pi\int_{-1}^{1}\left(x^2 - 4x + 3\right)dy$

$= \pi\int_{-1}^{1}\left(3 + 4y^2 + y^4\right)dy \therefore B$

Q6 $\ddot{x} = -g + kv^2 \Rightarrow v\dfrac{dv}{dx} = -g + kv^2 \Rightarrow \dfrac{dx}{dv} = \dfrac{v}{kv^2 - g}$

$\therefore x = \dfrac{1}{2k}\ln\left|kv^2 - g\right| + c$

$v = 0,\ x = 0 \therefore c = -\dfrac{1}{2k}\ln g$

$-2kx = \ln\left|\dfrac{g - kv^2}{g}\right| \Rightarrow g \times e^{-2kx} = g - kv^2$

$kv^2 = g - g \times e^{-2kx} \Rightarrow v^2 = \dfrac{g}{k}\left(1 - e^{-2kx}\right) \therefore A$

Q7 S on y axis $\therefore$ not C

$\dfrac{x^2}{1} + \dfrac{y^2}{8} = 1 \Rightarrow e^2 = 1 - \dfrac{a^2}{b^2} = \dfrac{7}{8} \Rightarrow be \neq 3$

$\dfrac{x^2}{4} + \dfrac{y^2}{5} = 1 \Rightarrow e^2 = 1 - \dfrac{a^2}{b^2} = \dfrac{1}{5} \Rightarrow be \neq 3$

$\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1 \Rightarrow e^2 = 1 - \dfrac{a^2}{b^2} = \dfrac{9}{25} \Rightarrow be = 5 \times \dfrac{3}{5} = 3 \therefore D$

Q8 $\sum\alpha^3 - 2\sum\alpha^2 + 9 = 0$

$\sum\alpha^3 = 2\sum\alpha^2 - 9 = 2\left[\left(\sum\alpha\right)^2 - 2\sum\alpha\beta\right] - 9$

$= 2[4-0] - 9 = -1 \quad \therefore B$

Q9 $y^2 - 4y + x = 0 \Rightarrow y = \dfrac{4 \pm \sqrt{16-4x}}{2} = 2 \pm \sqrt{4-x}$

$\therefore y_1 - y_2 = 2\sqrt{4-x} \Rightarrow A = \sqrt{4-x} \times 2\sqrt{4-x}$

$\Rightarrow \delta V = 2(4-x)\delta x \Rightarrow V = \int_0^4 2(4-x)\,dx \quad \therefore C$

Q10 $y = \dfrac{c^2}{x} \Rightarrow y' = -\dfrac{c^2}{x^2}$ @ $x = cp \Rightarrow y' = m_{\text{tang}} = \dfrac{-1}{p^2} \Rightarrow m_{norm} = p^2$

Normal is $y - \dfrac{c}{p} = p^2(x - cp) \Rightarrow p^2x - y - cp^3 + \dfrac{c}{p}$

$\Rightarrow p^3x - py + c - cp^4 = 0 \quad \therefore B$

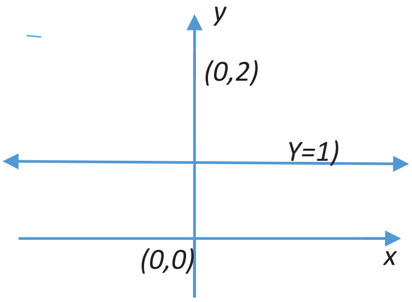
11a)i) $z = (1-i)^{-1} = \dfrac{1}{1-i} \times \dfrac{1+i}{1+i} = \dfrac{1}{2} + \dfrac{1}{2}i$

$\therefore \bar{z} = \dfrac{1}{2} - \dfrac{1}{2}i = \dfrac{1}{\sqrt{2}} cis\left(-\dfrac{\pi}{4}\right)$

ii) $(\bar{z})^{13} = \dfrac{1}{64\sqrt{2}} cis\left(-\dfrac{13\pi}{4}\right) = \dfrac{1}{64\sqrt{2}} cis\left(\dfrac{3\pi}{4}\right) = \dfrac{1}{64\sqrt{2}}\left(-\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right) = \dfrac{-1}{128} + i\dfrac{1}{128}$

b) $|z - 2i| = |z| \Rightarrow$ locus is all points equidistant from (0,0) and (0,2)

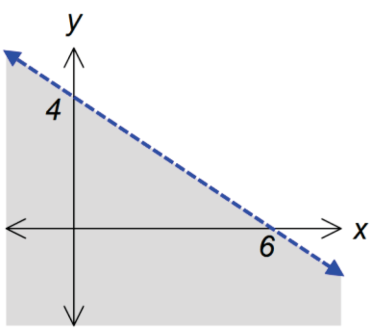
ie $y = 1$



c) $\Re e\left[(2-3i)z\right] < 12$

$(2-3i)(x+iy) = 2x + 3y + i(2y - 3x)$

$\therefore 2x + 3y < 12$



d) $x^3-3x^2-x+2=0\Rightarrow\sum\alpha=3$

$\therefore 2\alpha+\beta+\gamma=\alpha+\sum\alpha=\alpha+3$ and similarly for the other roots

$\therefore$ replace x with $y-3\Rightarrow(y-3)^3-3(y-3)^2-(y-3)+2=0$

$\Rightarrow y^3-9y^2+27y-27-3y^2+18y-27-y+3+2=0$

$\Rightarrow y^3-12y^2+44y-49=0$

e) $\dfrac{x^2+x+2}{(x^2+1)(x+1)}=\dfrac{Ax+B}{x^2+1}+\dfrac{C}{x+1}\Rightarrow x^2+x+2=(Ax+B)(x+1)+C(x^2+1)$

let $x=-1\Rightarrow 2=2C\therefore C=1$

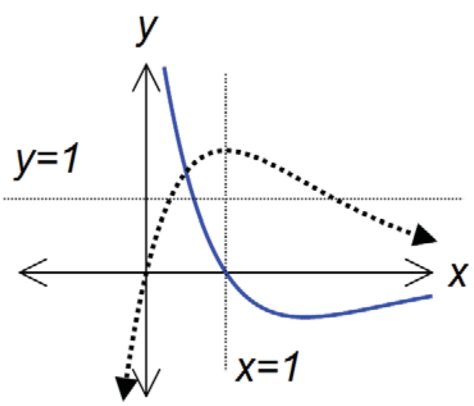
let $x=0\Rightarrow 2+B+C\Rightarrow B=1$

coefficients of $x^2\Rightarrow 1=A+C\therefore A=0$

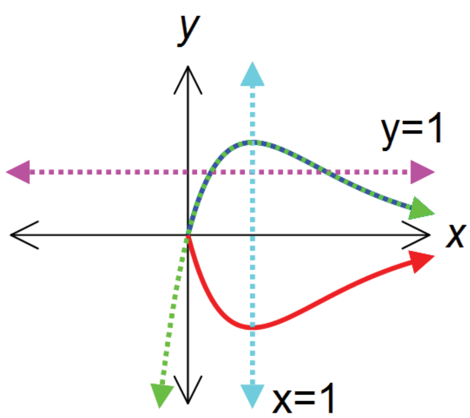
$\dfrac{x^2+x+2}{(x^2+1)(x+1)}=\left(\dfrac{1}{x^2+1}+\dfrac{1}{x+1}\right)$

ii) $\displaystyle\int\frac{x^2+x+2}{(x^2+1)(x+1)}dx=\int\left(\frac{1}{x^2+1}+\frac{1}{x+1}\right)dx=\tan^{-1}x+\ln|x+1|+d$

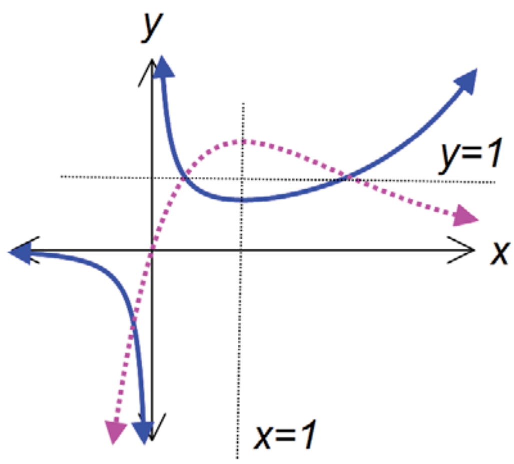
12a)i) $y=f'(x)$



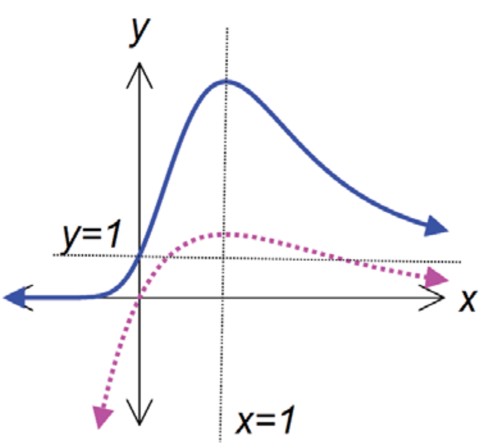
ii) $|y|=f(x)$



iii) $y=\dfrac{1}{f(x)}$



iv) $y=3^{f(x)}$



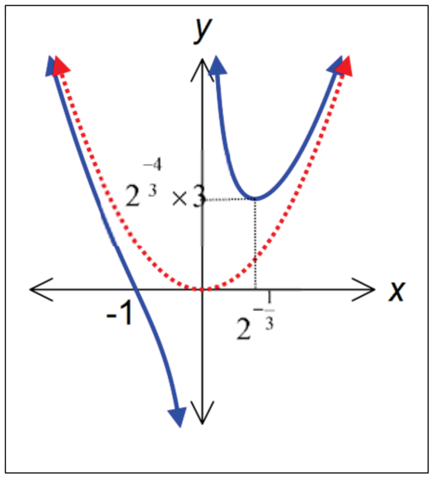
b) $\int_4^7 \frac{dx}{x^2-8x+19} = \int_4^7 \frac{dx}{(x-4)^2+3} = \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{x-4}{\sqrt{3}}\right)\right]_4^7 = \frac{1}{\sqrt{3}}\left[\tan^{-1}\sqrt{3} - \tan^{-1} 0\right] = \frac{\pi}{3\sqrt{3}}$

c)i) $f(x) = \frac{x^3+1}{x} = x^2 + \frac{1}{x} \Rightarrow f(x) - x^2 = \frac{1}{x} \therefore \lim_{x\to\infty}\left(f(x)-x^2\right) = \lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$

ii) $f'(x) = 2x - \frac{1}{x^2} = 0$ when $2x^3 = 1 \Rightarrow x = 2^{\frac{1}{3}}$

also $x \to 0^+ \quad f(x) \to \infty, \quad x \to 0^- \quad f(x) \to -\infty$

if $f(x) = 0 \Rightarrow x^3 = -1 \Rightarrow x = -1$



13a) $z^5 - 1 = 0 \Rightarrow (z-1)(z^4+z^3+z^2+z+1)$

if ω is complex $z - 1 \neq 0 \therefore z^4+z^3+z^2+z+1 = 0$

let $\alpha = \omega + \omega^4, \ \beta = \omega^2 + \omega^3 \ \therefore \alpha + \beta = -1$

$\alpha\beta = (\omega+\omega^4)(\omega^2+\omega^3) = \omega^3 + \omega^4 + \omega + \omega^2 = -1$

$\therefore$ quadratic is $z^2 - (\alpha + b)z + \alpha\beta = z^2 + z - 1 = 0$

b)i) $P(x) = x^3 + x^2 + mx + n$ since coefficients real if $z = 1 - 2i$ is a root so too is $z = 1 + 2i$

$\therefore \sum \alpha = 1 - 2i + 1 + 2i + \alpha = -1 \Rightarrow \alpha = -3$

$\therefore P(x) = (x+3)(x-1-2i)(x-1+2i)$

ii) $P(-3) = -27 + 9 - 3m + n = 0 \Rightarrow 3m - n = -18 \quad (1)$

$\prod \alpha = -3(1-2i)(1+2i) = -15 = -n \Rightarrow n = 15$

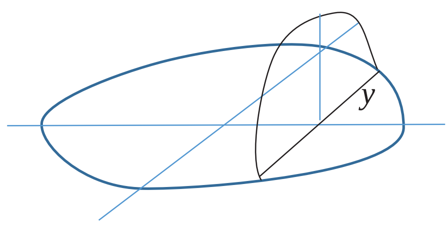
sub in (1) $\Rightarrow 3m = -18 + 15 \Rightarrow m = -1$

c)i) since $a = 6, b = 4 \Rightarrow \frac{x^2}{36} + \frac{y^2}{16} = 1$

ii) $A = \frac{\pi}{2}y^2 \Rightarrow \delta V = \frac{\pi}{2}y^2\delta x$

$\therefore V = \lim_{\delta x\to 0}\sum_{x=-6}^{6}\left(\frac{\pi}{2}y^2\delta x\right) = 2\times\frac{\pi}{2}\int_0^6 16\left(1-\frac{x^2}{36}\right)dx$

$= 16\pi\left[x - \frac{x^3}{108}\right]_0^6 = 16\pi\left(6 - \frac{216}{108}\right) = 64\pi \ U^3$



$d)i)$ let $P(x)=(x-\alpha)^3(x-\beta)$

$\therefore p'(x)=(x-\alpha)^3\times 1+3(x-\alpha)^2(x-\beta)$

$=(x-\alpha)^2[x-\alpha+3x-3\beta]$

$\therefore P'(x)$ has a zero of multiplicity 2

ii) $P(x)=8x^4-25x^3+27x^2-11x+1$

$P'(x)=32x^3-75x^2+54x-11$

$P''(x)=96x^2-150x+54$

$P''(1)=96-150+54=0$ and $P'(1)=32-75+54-11=0$

$\therefore x=1$ is a triple root $\Rightarrow P(x)=(x-1)^3(8x-1)$

$\therefore$ zeros are $1,\ 1,\ 1,\ \frac{1}{8}$

$14a)i)$ $\int_0^a f(a-x)\,dx=\int_0^a f(x)\,dx\ \therefore \int_0^\pi x\cos 2x\ dx=\int_0^\pi(\pi-x)\cos(2\pi-2x)\ dx=\int_0^\pi(\pi-x)\cos(2x)\ dx$

$\int_0^\pi x\cos 2x\ dx=\int_0^\pi \pi\cos 2x\ dx-\int_0^\pi x\cos 2x\ dx \Longrightarrow 2I=\pi\left[\frac{1}{2}\sin 2x\right]_0^\pi=0\ \therefore I=0$

$a)ii)$ $\delta V=\pi\left[(x+\delta x)^2-x^2\right]y$

$\approx 2\pi xy\delta x$

$V\approx \lim_{\delta x\to 0}\sum_{x=0}^{\pi}\delta V=\lim_{\delta x\to 0}\sum_{x=0}^{\pi}2\pi xy\delta x$

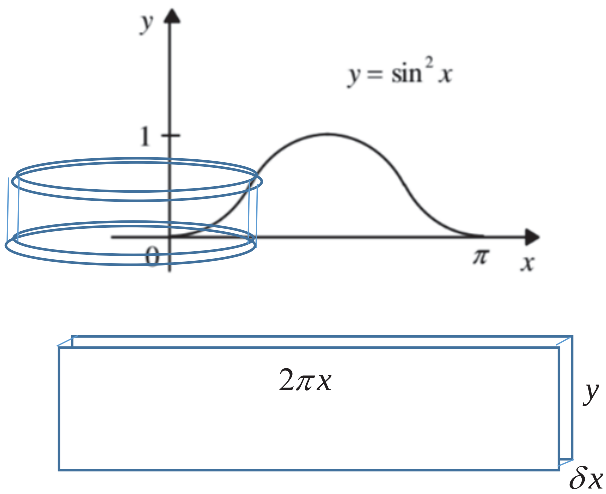
$=\lim_{\delta x\to 0}\sum_{x=0}^{\pi}2\pi x\sin^2 x\ \delta x$

$\therefore V=\pi\int_0^\pi x(1-\cos 2x)dx$

$=\pi\int_0^\pi x\ dx-\pi\int_0^\pi x\cos 2x\ dx$

$=\pi\left[\frac{x^2}{2}\right]_0^\pi-0$ from ii)

$=\frac{\pi^3}{2}$

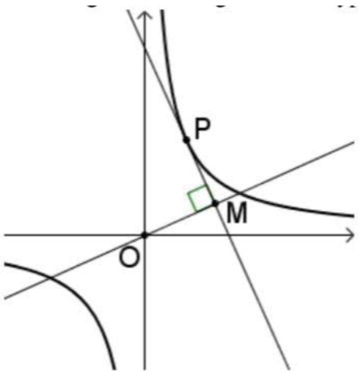


$b)i)$ $xy=1\Rightarrow y=\frac{1}{x}\Rightarrow y'=\frac{-1}{x^2}=\frac{-1}{t^2}$ at $x=t$

$\therefore$ tangent is $y-\frac{1}{t}=\frac{-1}{t^2}(x-t)$

$\Rightarrow x+t^2y=2t$ (1)

$ii)$ equation of OM $y-0=t^2(x-0)\Rightarrow y=t^2x$ (



iii) coordinates of M solve (1) with (2)

sub $y=t^2x$ in $x+t^2y=2t$

$$x+t^4x=2t \Rightarrow x=\frac{2t}{1+t^4},\ y=\frac{2t^3}{1+t^4}$$

from (2) $\frac{y}{x}=t^2 \Rightarrow t=\pm\sqrt{\frac{y}{x}}$ $\quad \therefore x=\frac{2\left(\pm\sqrt{\frac{y}{x}}\right)}{1+\frac{y^2}{x^2}}=\frac{2x^2\left(\pm\sqrt{\frac{y}{x}}\right)}{x^2+y^2}$

$$x\left(x^2+y^2\right)=2x^2\left(\pm\sqrt{\frac{y}{x}}\right) \quad \Rightarrow \quad x^2\left(x^2+y^2\right)^2=4x^4\frac{y}{x}$$

$x^2\left(x^4+2x^2y^2+y^4-4xy\right)=0$ reject $x^2=0$

$\therefore x^4+2x^2y^2+y^4-4xy=0$

Restrictions: $\frac{y}{x}>0$ for t to exist and $xy<1$ for M to lie outside H

c) $\ddot{x}=-g-kv \Rightarrow v\frac{dv}{dx}=-g-kv$

$$\frac{v}{g+kv}dv=-dx \Rightarrow \frac{\frac{1}{k}(kv+g)-\frac{g}{k}}{g+kv}=-dx$$

$$\int_V^0 \frac{1}{k}dv-\frac{g}{k}\int_V^0 \frac{1}{g+kv}dv=\int_0^H dx$$

$$\left[\frac{v}{k}-\frac{g}{k^2}\ln(g+kv)\right]_V^0=-H$$

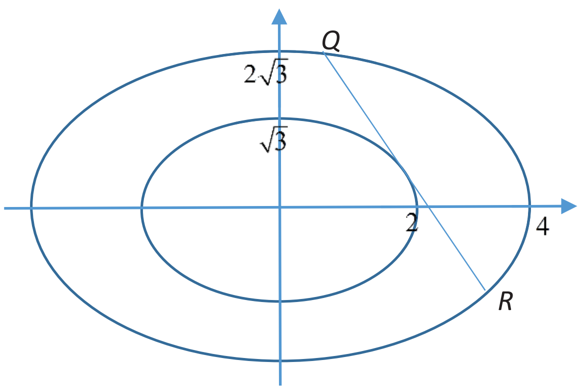
$$-\frac{g}{k^2}\ln g-\frac{V}{k}+\frac{g}{k^2}\ln(g+kV)=-H$$

$$H=\frac{V}{k}-\frac{g}{k^2}\ln\left(\frac{g+kV}{g}\right)=\frac{V}{k}-\frac{g}{k^2}\ln\left(1+\frac{kV}{g}\right)$$

15*a*) $\int_1^e x^7\ln x\,dx=\int_1^e \ln x\frac{d}{dx}\left(\frac{x^8}{8}\right)dx=\left[\frac{x^8}{8}\ln x\right]_1^4-\int_1^e \frac{x^8}{8}\times\frac{1}{x}\,dx$

$$=\frac{e^8}{8}-\frac{1}{8}\int_1^e x^7\,dx=\frac{e^8}{8}-\frac{e^8}{64}+\frac{1}{64}=\frac{7e^8+1}{64}$$

b)



i) $b^2=a^2\left(1-e^2\right) \Rightarrow \frac{3}{4}-1=-e^2$

$\Rightarrow e^2=\frac{1}{4} \Rightarrow e=\frac{1}{2}$

$\therefore$ Foci are $(\pm ae,0)$, $S,S'=(\pm 1,0)$

directrices are $x=\pm\frac{a}{e}=\pm\frac{2}{½}=\pm 4$

$ii)$ $\frac{x^2}{4}+\frac{y^2}{3}=1 \Rightarrow \frac{x}{2}+\frac{2y}{3}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-3x}{4y}=\frac{-6\cos p}{4\sqrt{3}\sin p}=-\frac{\sqrt{3}\cos p}{2\sin p}$ @ P

$\therefore$ Tangent at P is $y-\sqrt{3}\sin p=-\frac{\sqrt{3}\cos p}{2\sin p}(x-2\cos p)$

$2y\sin p-2\sqrt{3}\sin^2 p=-\sqrt{3}x\cos p+2\sqrt{3}\cos^2 p$

$\sqrt{3}x\cos p+2y\sin p=2\sqrt{3} \Rightarrow \frac{x\cos p}{2}+\frac{y\sin p}{\sqrt{3}}=1$

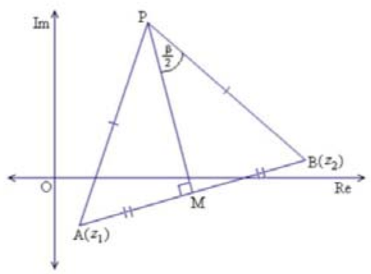
$iii)$ Let this tangent meet E_2 at $T\left(4\cos t, 2\sqrt{3}\sin t\right)$

$\therefore \frac{4\cos t\cos p}{2}+\frac{2\sqrt{3}\sin t\sin p}{\sqrt{3}}=1 \Rightarrow \cos t\cos p+\sin t\sin p=\frac{1}{2}$

$\therefore \cos(t-p)=\frac{1}{2} \Rightarrow t-p=\pm\frac{\pi}{3}$ since $0<p<\frac{\pi}{2}$ and $-\pi<t<\pi$

$\therefore$ if $q=p+\frac{\pi}{3}$ then $r=p-\frac{\pi}{3} \Rightarrow |q-r|=\frac{2\pi}{3}$

c)



$c)i)\alpha)$ $\overrightarrow{OA}+\overrightarrow{AM}=\overrightarrow{OM}$ or $\overrightarrow{AM}=\frac{1}{2}\overrightarrow{AB}=\frac{1}{2}(z_2-z_1)$

$\overrightarrow{AM}=\overrightarrow{OM}-\overrightarrow{OA}$

$=\frac{z_1+z_2}{2}-z_1=\frac{z_2-z_1}{2}$

$\beta)$ $\overrightarrow{MP}=\overrightarrow{AM}$ rotated through 90^o and increased in length by a factor $\cot\frac{\beta}{2}$

$\left(\text{since } \tan\frac{\beta}{2}=\frac{|AM|}{|PM|} \Rightarrow |PM|=\cot\frac{\beta}{2}|AM|\right)$

$\therefore \overrightarrow{MP}=i\left(\frac{z_2-z_1}{2}\right)\cot\frac{\beta}{2}$

$ii)$ $\overrightarrow{OP}=\overrightarrow{OM}+\overrightarrow{MP}=\frac{z_1+z_2}{2}+i\left(\frac{z_2-z_1}{2}\right)\cot\frac{\beta}{2}$

$=\frac{1}{2}z_1\left[1-i\cot\frac{\beta}{2}\right]+\frac{1}{2}z_2\left[1+i\cot\frac{\beta}{2}\right]$

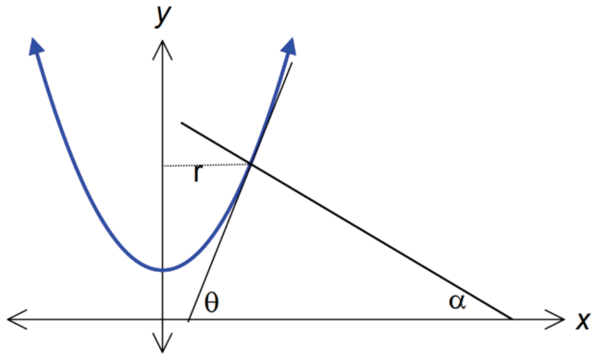
16a)i) $y^2 - x^2 = 1 \Rightarrow y = \sqrt{1+x^2}$

$$\Rightarrow y' = \frac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}} \times 2x = \frac{x}{\sqrt{1+x^2}}$$

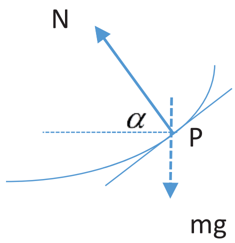
when $x = r$ $\quad y' = \tan\theta = \dfrac{r}{\sqrt{1+r^2}}$

but from diagram $\alpha = \dfrac{\pi}{2} - \theta$

$$\therefore \tan\alpha = \cot\theta = \frac{\sqrt{1+r^2}}{r}$$



ii)



iii) let the normal reaction be N

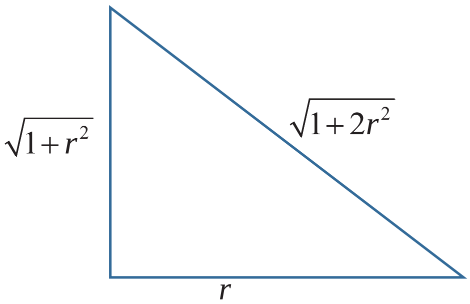
resolve forces vetically and horizontally

V: $N\sin\alpha = mg$ (1)

$H: N\cos\alpha = mr\omega^2$ (2)

$$(1) \div (2) \Rightarrow \tan\alpha = \frac{g}{r\omega^2} = \frac{\sqrt{1+r^2}}{r}$$

$$\therefore \frac{g}{\omega^2} = \sqrt{1+r^2} \Rightarrow r^2 = \frac{g^2}{\omega^4} - 1$$



$$\text{sub in (1)} \Rightarrow N = \frac{mg}{\sin\alpha} = mg \div \frac{\sqrt{1+r^2}}{\sqrt{1+2r^2}} = mg \div \frac{\frac{g}{\omega^2}}{\sqrt{1+2\left(\frac{g^2}{\omega^4} - 1\right)}}$$

$$= mg \times \frac{\sqrt{\frac{2g^2}{\omega^4} - 1}}{\frac{g}{\omega^2}} = m\sqrt{2g^2 - \omega^4}$$

iv) from H $\quad y = \sqrt{1+x^2}$ when $x = r \Rightarrow y = \sqrt{1+r^2}$

from iii) $r^2 = \dfrac{g^2}{\omega^4} - 1 \Rightarrow y = \sqrt{\dfrac{g^2}{\omega^4}} = \dfrac{g}{\omega^2}$

but $1 \le y \le 5$ $\therefore$ $\quad 1 \le \dfrac{g}{\omega^2} \le 5 \Rightarrow \dfrac{1}{5} \le \dfrac{\omega^2}{g} \le 1$

$$\Rightarrow \sqrt{\frac{g}{5}} \le \omega \le \sqrt{g}$$

16(b) Let $I_n = \displaystyle\int_1^e (1 - \ln x)^n \, dx$ where $n = 0, 1, 2, \ldots$

(i) Show that $I_n = -1 + nI_{n-1}$ where $n = 0, 1, 2, \ldots$ **2**

$$I_n = \int_1^e (1-\ln x)^n\, dx = \int_1^e (1-\ln x)^n \frac{d}{dx}(x)\, dx$$
$$= \left[x(1-\ln x)^n\right]_1^e - \int_1^e nx(1-\ln x)^{n-1} \times \frac{-1}{x}\, dx$$
$$= -1 + nI_{n-1}$$

(ii) Hence evaluate $\int_1^e (1-\ln x)^3\, dx$ **2**

$$I_3 = -1 + 3I_2 = -1 + 3(-1 + 2I_1) = -1 - 3 + 6(-1 + I_0)$$
$$= -10 + 6\int_1^e dx = -10 + 6e - 6 = 6e - 16$$

(iii) Show that $\frac{I_n}{n!} = e - \sum_{r=0}^{n} \frac{1}{r!}$ where $n = 1, 2, 3, ...$ **2**

$$\frac{I_n}{n!} = \frac{-1 + nI_{n-1}}{n!} = \frac{-1}{n!} + \frac{I_{n-1}}{(n-1)!} = \frac{-1}{n!} + \frac{-1 + (n-1)I_{n-2}}{(n-1)!} = \frac{-1}{n!} + \frac{-1}{(n-1)!} + \frac{I_{n-2}}{(n-2)!}$$
$$= \frac{-1}{n!} + \frac{-1}{(n-1)!} + \frac{-1 + (n-2)I_{n-3}}{(n-2)!} = \frac{-1}{n!} + \frac{-1}{(n-1)!} + \frac{-1}{(n-2)!} + \frac{I_{n-3}}{(n-3)!}$$
$$= \frac{-1}{n!} + \frac{-1}{(n-1)!} + \frac{-1}{(n-2)!} + \ldots + \frac{-1}{1!} + \frac{I_0}{0!}$$
$$= -\sum_{r=1}^{n} \frac{1}{r!} + \int_1^e dx = -\sum_{r=1}^{n} \frac{1}{r!} + [x]_1^e = -\sum_{r=1}^{n} \frac{1}{r!} + e - 1 = -\sum_{r=1}^{n} \frac{1}{r!} + e - \frac{1}{0!} = -\sum_{r=0}^{n} \frac{1}{r!} + e$$

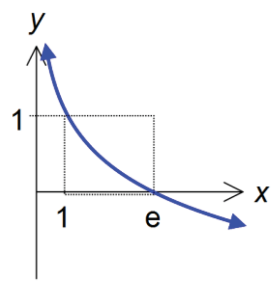
(iv) Show that $0 \le I_n \le e - 1$ **1**

Consider $y = 1 - \ln x$ for $1 \le x \le e \Rightarrow 0 \le y \le 1$

$\therefore$ for $y = (1-\ln x)^n$ $\quad 0 \le y \le 1$ also

clearly $\int_1^e (1-\ln x)^n\, dx \ge 0$ but $\le$ rectangle $= 1 \times (e-1)$

$\therefore 0 \le I_n \le (e-1)$



(v) Deduce that $\lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{r!} = e$ **1**

from (iv) $0 \le \frac{I_n}{n!} \le \frac{e-1}{n!}$

$\therefore n \to \infty$ $\quad \frac{e-1}{n!} \to 0 \quad \therefore 0 \le \lim_{n\to\infty} \frac{I_n}{n!} \le 0$

using (iii) $\lim_{n\to\infty}\left(e - \sum_{r=0}^{n} \frac{1}{r!}\right) = 0 \Rightarrow e - \lim_{n\to\infty}\left(\sum_{r=0}^{n} \frac{1}{r!}\right) = 0 \Rightarrow e = \lim_{n\to\infty}\left(\sum_{r=0}^{n} \frac{1}{r!}\right)$